ELECTRICAL PARAMETERS OF A CHANNEL WITH FINITE ELECTRODES WITH ALLOWANCE FOR THE ELECTRODE POTENTIAL DROP

A. B. Vatazhin

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 7, No. 6, pp. 63–66, 1966

Spatial problems involving the electric field in an MHD channel were formulated in [1] with allowance for the electrode potential drop. It was assumed that the electrode layer had a small thickness, so that relationships on the boundary of the layer could be applied to the surface of the electrode. It was assumed that the electrode potential drop $\delta\varphi^\circ$ could be represented as a function of the current density j_n° at the electrode in the form of a known function $\delta\varphi^\circ = f(j_n^\circ)$ determined experimentally or deduced from the appropriate electrode-layer theory. An approximate method was then put forward for solving such problems by reducing them to the determination of the electric field from a known distribution of the magnetic field and the gas-dynamic parameters. It was shown that when $\varepsilon = \delta\varphi^\circ/E$ is small (E is the characteristic induced or applied potential difference), the solution can be sought in the form of series in powers of ε. In the zero-order approximation, the electric field is determined without taking into account the electrode processes. The first approximation gives a correction of the order of ε. The quantity $\delta\varphi^\circ$, which is present in the boundary conditions on the electrode in the first-order approximation, is determined from the current density calculated in the zero-order approximation.

One of the problems discussed in [1] was concerned with the electric current in a channel with one pair of symmetric electrodes. Its solution was found in the first approximation in the form of the integral Keldysh-Sedov formula. In this paper we report an analysis of the solution for $\delta\varphi^\circ$ taken in the form of a step function.

Consider a channel of height h with two symmetric electrodes of length 2λ (Fig. 1), and suppose that a medium having a constant conductivity σ in the magnetic field $B^\circ = 0, 0 - B^\circ(x^\circ)$ flows along the channel with velocity $v^\circ = V^\circ(x^\circ), 0, 0$. The vectors v° and B° are assumed to be known [2]. As a result of the interaction between the field and the medium, a potential difference $\varphi^{\circ+} - \varphi^{\circ-}$ is induced between the electrodes (we shall assume that this difference is given), and a current J° (which is to be determined) flows through the load connecting the electrodes. The distribution of the electric current I° and potential φ° in the channel for isotropic conductivity and even function $B^\circ V^\circ$ can be found from the following set of equations [1]:

$$j_x = -\partial\varphi/\partial x, \qquad j_y = -\partial\varphi/\partial y + q(x),$$
$$\Delta\varphi = 0 \qquad (q = B^\circ V^\circ),$$
$$\partial\varphi/\partial y = q \quad \text{on } CD,\ MD,$$
$$\partial\varphi/\partial x = 0 \ \text{on } FOE,$$
$$\varphi = \varphi^+ + \delta\varphi^+(j_y) \quad \text{on } FM,$$
$$\varphi = \varphi^- + \delta\varphi^-(-j_y) \quad \text{on } EC. \tag{1}$$

All the quantities in Eq. (1) are dimensionless. The velocity, the magnetic field, the density of the electric current, the potential, and the coordinates are referred to V^*, B^*, $c^{-1}\sigma B^* V^*$, $c^{-1} h B^* V^*$ and h, respectively. The functions $\delta\varphi^+(j_y)$ and $\delta\varphi^-(-j_y)$ (whose form is assumed known) determine the electrode potential drop across FM and EC. When the magnetic field is inhomogeneous, reverse-current zones appear on the electrodes (usually near their ends), and the electrode surface consists of individual regions, some of which operate under cathode conditions, while others act as anodes. The form of $\delta\varphi$ is different for different regions of this kind. Boundary currents on these regions must be determined by solving the set of equations given by Eq. (1). In many cases, the electrode potential drop is relatively small: $\delta\varphi = \varepsilon\, s(j_n)$, $\varepsilon = o(1)$, $s = O(1)$. The solution of Eq. (1) can then be sought in the form of the series

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + \ldots,$$
$$j_x = j_{x0} + \varepsilon j_{x1} + \ldots, \quad j_y = j_{y0} + \varepsilon j_{y1} + \ldots \tag{2}$$

The corresponding solutions for the zero-order approximation were obtained in [3, 4].

The set of equations for the first approximation and its solution are of the form

$$j_{x1} = -\frac{\partial\varphi_1}{\partial x}, \quad j_{y1} = -\frac{\partial\varphi_1}{\partial y}, \qquad \Delta\varphi_1 = 0$$
$$\frac{\partial\varphi_1}{\partial y} = 0 \quad \text{on } CD,\ MD;$$
$$\frac{\partial\varphi_1}{\partial x} = 0 \quad \text{on } FOE, \quad \varphi_1 = s_0^+ \quad \text{on } FM,$$
$$\varphi_1 = s_0^- \quad \text{on } EC \qquad (s_0^+ = s\,[j_{y0}(x, {}^1/_2)],$$
$$s_0^- = s\,[-j_{y0}(x, -{}^1/_2)]), \tag{3}$$

$$w(z) = \frac{\partial\varphi_1}{\partial x} - i\frac{\partial\varphi_1}{\partial y} = \frac{1}{\pi g(t)}\int_k^1\left[\left(\frac{1-\rho}{1+\rho}\right)^{1/2}\frac{\beta^-(\rho)}{\rho - t} - \right.$$
$$\left. - \left(\frac{1+\rho}{1-\rho}\right)^{1/2}\frac{\beta^+(\rho)}{\rho + t}\right]d\rho + \frac{\gamma}{\sqrt{(t-1)(t+1)}},$$
$$z = x + iy, \qquad t = k\sin\pi i z,$$
$$g(t) = \left(\frac{t-1}{t+1}\right)^{1/2}, \qquad \gamma = \frac{\pi r - i_4}{2K(k)}, \quad k = \operatorname{sch}\frac{\pi\lambda}{h},$$
$$r = s_0^+ - s_0^- \quad \text{for } x = 0, \qquad \beta^+(\rho) = \frac{ds_0^+}{dx},$$
$$\beta^-(\rho) = \frac{ds_0^-}{dx} \quad \text{for } x = x(\rho) = \frac{1}{\pi}\operatorname{ar\,ch}\frac{\rho}{k},$$
$$i_4 = \frac{1}{\pi}\int_{-k}^{k}\left(\frac{1+\tau}{(1-\tau)(k^2-\tau^2)}\right)^{1/2}\left\{\int_k^1\left[\left(\frac{1-\rho}{1+\rho}\right)^{1/2}\frac{\beta^-(\rho)}{\rho-\tau} - \right.\right.$$
$$\left.\left. - \left(\frac{1+\rho}{1-\rho}\right)^{1/2}\frac{\beta^+(\rho)}{\rho+\tau}\right]d\rho\right\}d\tau, \tag{4}$$

where K(k) is the complete elliptic integral of the first kind, and for the square roots we have taken the branches which are positive for $t = \tau > 1$. The function $t(z)$ provides the conformal representation of the right-hand half of the channel on the upper half-plane.

The electric current J_1 and the work A_1 done by the medium in overcoming the resistance offered by the magnetic field are given by

$$J_1 = 2\int_0^b j_{y1}(x, {}^1/_2)\,dx =$$

$$= \frac{1}{\pi}\left[-\pi r\alpha^* + i_4\alpha^* + 2(i_1 - i_2 - i_3)\right], \qquad (5)$$

$$(\alpha^* = K(k')/K(k), \quad k^2 + k'^2 = 1, \quad b = \lambda/h)$$

$$i_1 = \frac{1}{\pi}\int_k^1 \left(\frac{1+\tau}{(1-\tau)(\tau^2-k^2)}\right)^{1/2}\left[\int_k^1 \left(\frac{1+\rho}{1-\rho}\right)^{1/2} \frac{\beta^+(\rho)\,d\rho}{\rho+\tau}\right] d\tau,$$

$$i_2 = \frac{1}{\pi}\int_k^1 \ln\frac{1-\tau}{\tau-k}\,\frac{\beta^-(\tau)\,d\tau}{\sqrt{\tau^2-k^2}},$$

$$i_3 = \frac{1}{\pi}\int_k^1 \left(\frac{1+\tau}{(1-\tau)(\tau^2-k^2)}\right)^{1/2}\left\{\int_k^1\left[\left(\frac{1-\rho}{1+\rho}\right)^{1/2}\beta^-(\rho) - \right.\right.$$

$$\left.\left. -\left(\frac{1-\tau}{1+\tau}\right)^{1/2}\beta^-(\tau)\right]\frac{d\rho}{\rho-\tau}\right\} d\tau,$$

$$A_1 = 2\int_0^\infty \int_{-1/2}^{1/2} q j_{y1}\,dx\,dy =$$

$$= 2\int_0^b q(s_0^- - s_0^+)\,dx + 2\int_b^\infty q\nu\,dx,$$

$$\nu(x) = s_0^-(b, -1/2) - s_0^+(b, 1/2) + \int_b^x \mu\,dx,$$

$$\mu(x) = \frac{2\tau}{\sqrt{\tau^2-1}} - \frac{2}{\pi\sqrt{\tau^2-1}} \times$$

$$\times \int_k^1 \left[\left(\frac{1-\rho}{1+\rho}\right)^{1/2}\frac{\beta^-(\rho)(\tau^2+\rho)}{\tau^2-\rho^2} + \right.$$

$$\left. + \left(\frac{1+\rho}{1-\rho}\right)^{1/2}\frac{\beta^+(\rho)(\tau^2-\rho)}{\tau^2-\rho^2}\right] d\rho \quad (\tau = k\,\mathrm{ch}\,\pi x). \qquad (6)$$

Calculations based on Eqs. (5) and (6) are exceedingly difficult. However, if the electrode potential drop can be regarded as constant (this is valid for sufficiently high current densities and metal electrodes [5]), the calculations are considerably simpler.

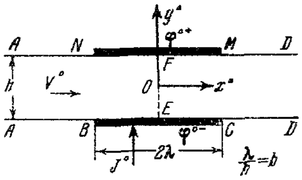

Fig. 1

We shall assume that the anode potential difference (which is usually much smaller than the cathode potential difference) is equal to zero. The functions s_0^+ and s_0^- and their derivatives are then given by

$$s_0^+ = \begin{cases}1 & \text{for } 0 < x < x^*, \\ 0 & \text{for } x^* < x < b,\end{cases} \qquad s_0^- = \begin{cases}0 & \text{for } 0 < x < x^* \\ 1 & \text{for } x^* < x < b\end{cases},$$

$$ds_0^+/dx = -\delta(x - x^*), \quad ds_0^-/dx = \delta(x - x^*), \qquad (7)$$

where $\delta(x)$ is the delta function. The point x^* on the upper (lower) electrode separates the cathode (anode) region $(0, x^*)$ from the anode (cathode) region (x^*, b) and is determined from the solution in the zero-order approximation.

Subject to the conditions given by Eq. (7), the integrals i_ν ($\nu = 1, \ldots, 4$) and external current J_1 are given by

$$i_1 = -\left(\frac{(1+a)(a^2-k^2)}{1-a}\right)^{1/2}\int_k^1\left(\frac{1+\tau}{(1-\tau)(\tau^2-k^2)}\right)^{1/2}\frac{d\tau}{a+\tau},$$

$$i_2 = \ln\frac{1-a}{a-k},$$

$$i_3 = -i_2 + \left(\frac{(a^2-k^2)(1-a)}{1+a}\right)^{1/2}\int_k^1\left(\frac{1+\tau}{(1-\tau)(\tau^2-k^2)}\right)^{1/2}\frac{d\tau}{a-\tau},$$

$$i_4 = 4a\chi i_5, \quad i_5 = \int_3^k\left(\frac{1-\tau^2}{k^2-\tau^2}\right)^{1/2}\frac{d\tau}{a^2-\tau^2},$$

$$J_1 = \frac{1}{\pi}\left[-\pi\alpha^* + 4a\chi(\alpha^* i_5 - i_6)\right],$$

$$i_6 = \int_k^1\left(\frac{1-\tau^2}{\tau^2-k^2}\right)^{1/2}\frac{d\tau}{a^2-\tau^2} \qquad \left(\chi = \left(\frac{a^2-k^2}{1-a^2}\right)^{1/2},\right.$$

$$\left. k \leqslant a = k\,\mathrm{ch}\,\pi x^* \leqslant 1\right), \qquad (8)$$

We note that, subject to Eq. (7), the current density on the electrodes at x = x* becomes infinite, and the

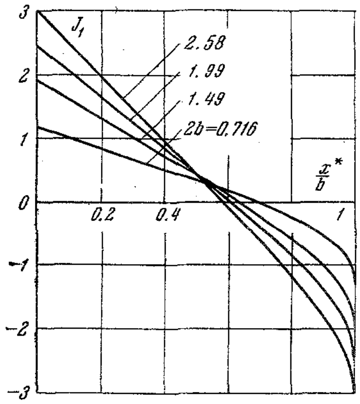

Fig. 2

integral i_6 diverges. This occurs because the potential is given by a discontinuous function whereas, strictly speaking, it should be continuous along the channel walls. Near the point x = x* the functions can no longer be represented by the series of Eq. (2). However, integral quantities such as the current J_1 can be determined from the above formulas by considering the corresponding integrals (for example, i_6) as singular (in Cauchy's sense), and evaluating their principal values.

Consider the current J_1. The principal values of the integral i_6 and the integral i_5 are given by

$$i_6 = K(k') + \frac{1}{\chi}\left[\frac{1}{2a}\ln\frac{(1+a)(a-k)}{(1-a)(a+k)} - \right.$$

$$\left. - \int_0^{\pi/2}\frac{\mathrm{ctg}(\nu+\nu_*)\,d\nu}{\sqrt{1-k'^2\sin^2\nu}}\right],$$

$$i_5 = K(k) + \frac{1-a^2}{a^2}\Pi\left(\frac{\pi}{2}, -\frac{k^2}{a^2}, k\right)$$

$$(k'\sin\nu_* = \sqrt{1-a^2}), \qquad (9)$$

where Π is the complete elliptic integral of the third kind. By going to the limit as $a \to k$ and $a \to 1$ in Eq. (8), and taking Eq. (9) into account, we find that $J_1 = \alpha^*$ for $x^* = 0$, and $J_1 = -\alpha^*$ for $x^* = b = \lambda/h$.

It is readily seen that these results will be the solutions of the well-known problem of the distribution of current in a channel with unit potential difference between the electrodes [6, 7].

Figure 2 shows a plot of the function $J_1(x^*/b)$ with $2b = 2\lambda/h$ as a parameter. If the upper electrode operates as a cathode, then $x^*/b = 1$

and the current losses due to electrode effects are at a maximum. When the reverse-current zones appear near the ends of the electrodes, these losses are found to decrease. For each b there is a range $0 \le x^* < x^{**} < b$, for which $J_1 > 0$. Therefore, if the end effect is very strongly defined (the reverse-current area of the electrode is considerable), the electrode processes, for which Eq. (7) is valid, may lead to an increase in the external current. However, we must remember that from a practical standpoint the important values of x^*/b are those for which most of the upper (lower) electrodes operates as a cathode (anode), since otherwise the resultant characteristics of the device (determined largely in the zero-order approximation) turn out to be very low.

The fact that $J_1 \neq 0$ for $x^* = b/2$ is explained by the presence of edge effects. If the condition $j_x = 0$ is satisfied on the lines NB and MC, then $J_1(1/2) = 0$. As b increases, the influence of the edge effects is reduced. Therefore, the point x^{**} at which $J_1 = 0$ approaches $x^* = b/2$.

REFERENCES

1. A. B. Vatazhin, "Electric fields in magnetohydrodynamic channels in the presence of electrode potential drop," PMM, vol. 30, no. 3, 441–450, 1966.

2. A. B. Vatazhin and S. A. Regirer, "Approximate calculation of the current distribution in the flow of a conducting medium along a channel in a magnetic field," PMM, vol. 26, no. 3, 548–556, 1962.

3. G. W. Sutton, H. Hurwitz Jr., and H. Poritsky, "Electrical and pressure losses in a magnetohydrodynamic channel due to end current loops," Commun. and Electronics, no. 5, 687–695, 1962.

4. A. B. Vatazhin, "Magnetohydrodynamic flow in a plane channel with finite electrodes," Izv. AN SSSR, OTN, Mekhanika i mashinostroenie, no. 1, 52–58, 1962.

5. G. A. Lyubimov, "Electrode layers on hot electrodes," PMTF [Journal of Applied Mechanics and Technical Physics], no. 4, 45–53, 1965.

6. K. K. Tabaks, Determination of the Electric Field in a DC Electromagnetic Pump [in Russian], Uch. zap. Latv. un-ta, vol. 21, 1958.

7. A. B. Vatazhin, "Determination of Joule dissipation in the channel of a magnetohydrodynamic generator," PMTF, no. 5, 59–69, 1962.

1 September 1966 Moscow